Dr. Marques Sophie
Office 519

Linear algebra

Fall Semester 2013

marques@cims.nyu.edu

Problem Set # 6

Exercise 1($\star$):

1. Verify that $W = \{z \in \mathbb{C}^5 : z_2 = z_3 = z_4 \text{ and } z_1 + z_5 = 0\}$ is a vector space. Then describe W and its dimension by producing a basis for this subspace.
2. Verify that $W = \{z \in \mathbb{C}^n : z_1 + z_2 + ... + z_n = 0\}$ is a subspace of $V = \mathbb{C}^n$, determine its dimension and exhibit a basis.

Exercise 2($\star$):
If V is finite dimensional vector space over K and W a vector subspace such that $dim_K(V) = dim_K(W)$, explain why we must have $V = W$.

Exercise 3($\star$):
We define the trace $Tr(A)$ of an $n \times n$ matrix to be $Tr(A) = \sum_{i=1}^{n} A_{ii}$ (sum of the diagonal values). As we shall see, this easily computed "invariant" incorporates important information about the structure of the operator. Prove the following properties of the trace operator $Tr : M_n(K) \to K$

1. Tr is a linear operator;
2. $Tr(AB) = Tr(BA)$, for all pairs A, $B \in M_n(K)$ (even if $AB \neq BA$);
3. If U is an invertible matrix then $Tr(UAU^{-1}) = Tr(A)$, for all $A \in M_n(K)$.

Exercise 4($\star$):
Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear operator such that $T(e_1) = 0$ and $T(e_2) = e_1$ (so its matrix with respect to the standard basis $\mathcal{X} = \{e_1, e_2\}$ is $[T]_{\mathcal{X}\mathcal{X}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Prove that no basis $\mathcal{Y} = \{f_1, f_2\}$ can make $[T]_{\mathcal{Y}\mathcal{Y}}$ diagonal.
Hint: You can use without proving it that $S = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is invertible if and only if the determinant of S $det(S) = a_{11}a_{22} - a_{21}a_{12}$ is nonzero.

Exercise 5($\star$):

1. Prove that the transpose 0^t of the zero operator $0(v) = 0_W$ from $V \to W$ is the zero operator from $W^* \to V^*$, so $0^t(l) = 0_{V^*}$, for all $l \in W^*$.

2. If $V = W$ and $Id_V : V \to V$ is the identity map $Id_V(v) = v$, for any v. Prove that $(Id_V)^t$ is the identity map $Id_{V^*} : V^* \to V^*$.

3. Prove that $(\lambda_1 T_1 + \lambda_2 T_2)^t = \lambda_1 T_1^t + \lambda_2 T_2^t$, for any $\lambda_1, \lambda_2 \in K$ and $T_1, T_2 : V \to W$.

4. Let $V \xrightarrow{S} W \xrightarrow{T} M$ be linear maps between finite dimensional vector spaces. Then prove:
$$(T \circ S)^t = S^t \circ T^t$$
Note the reversal of order when we compute the transpose of a product.

5. If V, W are finite dimensional and $T : V \to W$ is an invertible linear operator (i.e. a bijection), prove that $T^t : W^* \to V^*$ is invertible too and $(T^{-1})^t = (T^t)^{-1}$ as maps from $V^* \to W^*$.